

Towards $c = 0$ flows

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Abstract

We discuss some implications of the gravitational dressing of the renormalization group for conformal field theories perturbed by relevant operators. The renormalization group flows are defined with respect to the dilatation operator associated with the $J_0^{(0)}$ mode of the $SL(2, R)$ affine algebra. We discuss the possibility of passing under the $c = 25$ barrier along renormalization group flows in some models.

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1 Introduction

Polyakov's fundamental string is presented as a certain two dimensional conformal field theory (CFT) coupled to two dimensional quantum gravity [1]. One of the physical conditions that has to be implemented in such theories is the vanishing of the total Virasoro central charge of the system [2]. However this condition alone is not sufficient to pick out a single string model. In fact, there are a large number of such non-critical strings all of which obey the given physical criterion. It is widely believed that string field theory will provide us with new principals for selecting the true ground state of string theory. Although this theory is still being developed we can already make out some of its basic features.

One of the most important notions in string field theory ought perhaps to be the independence of its action on any particular CFT describing a certain string vacuum. At present, we know how the string field action can be obtained in the vicinity of a given CFT [3]. It has been observed that this action is in fact invariant under any truly marginal perturbation of the given conformal background [4],[5]. Such perturbations may amount to topology change in the corresponding target manifolds. In particular, it has been shown now by performing truly marginal deformations one can connect two string solutions corresponding to two different Calabi-Yau manifolds [8]. This result elucidates a certain background independence of the string field action.

Of course in order to talk at all about the vicinity of a CFT, one has to define the meaning of the distance in the space of models. In general, this is a difficult question to answer. However, there does exist such a definition for marginal deformations [4], [5]. One can think of the space of models as a space parametrized by coupling constants, and variations in the space of models amount to variations of these coupling constants. It is well known that one can arrive at a new conformal theory by doing relevant perturbations on a given one [9]. Thus, it might be interesting to check the covariance of string field theory under relevant perturbations. This requires the definition of a metric and connection on the space of couplings (see e.g. [6], [7]).

By relevant perturbations we mean deformations of a CFT by operators whose con-

formal dimensions are less than one. In string field theory such a perturbed system is to be considered coupled to 2d gravity. Because relevant perturbations break the conformal symmetry of the matter system, the treatment of the perturbed model is different from the case of a CFT perturbed by (truly) marginal operators in the presence of 2d gravity. This interaction between the quantum gravity and the nonconformal matter system leads to an interacting theory which must be conformally invariant. There will be constraints that arise as a consequence of the conformal symmetry in the interacting system. However, in this conformal system there is no longer a sensible splitting into conformal matter and $SL(2, R)$ invariant quantum gravity. Therefore, the gravitational dressing becomes an even more nontrivial issue in such theories.

The aim of the present paper is to investigate renormalization group flows in CFT's perturbed by relevant operators in the presence of 2d gravity. We would like to point out that the renormalization group in the presence of quantum gravity has been extensively studied during the last few years [11]-[21]. Our attention will focus on effects which occur when the quantum gravity couples to the CFT perturbed by relevant quasimarginal operators. We will show in the context of particular matter systems, that flows under the $c = 25$ barrier are possible. From the point of view of the results obtained in [10], flows under the $c = 25$ barrier should result in changing minkowskian string solutions into euclidean string solutions. We will argue that such flows do not contradict the basic notions of two dimensional quantum field theory.

Instead of working in the conformal gauge, we shall use the light cone gauge in which there is a definition of the dilatation operator due to Polyakov et al [2],[17]. Scale transformations of the renormalization group are defined with respect to this operator within the perturbation expansion in the coupling constant. From this, one can calculate the gravitational dressing of the beta function. Given this beta function we can obtain a generalization of the Cardy-Ludwig formula for the difference between Virasoro central charges of the matter at IR and UV fixed points of the flow [22] in the presence of 2d gravity. Based on this formula one can exhibit flows between $c > 25$ and $c < 25$ conformal models coupled to 2d gravity.

The paper is organized as follows. In section 2 relevant perturbations of CFT's in the

presence of 2d gravity are considered. In section 3 we explore the effect of the dressing of the renormalization group due to 2d quantum gravity. Finally, in the last section we discuss the possibility of R.G. flows under the $c = 25$ barrier. As an example to illustrate this, we investigate the nonunitary WZNW model on the group manifold G , with $\dim G = 25$.

2 Relevant perturbations in the presence of quantum gravity

Given a CFT one can decide whether or not there is an operator O with following properties: that its conformal dimension is less than one and that the operator product expansion with itself obeys

$$O O = [O] + [I] + \dots \quad (2.1)$$

where I is the identity operator. The square brackets denote the contributions of the corresponding operators and their descendants, whereas dots stand for operators with scaling dimensions greater than one. If such an operator exists in the spectrum of the theory defined by an action S , then one can consider another model with action

$$S(\epsilon) = S - \epsilon \int d^2x O(x). \quad (2.2)$$

If the coupling ϵ is small, then $S(\epsilon)$ can be treated as a perturbation of S . The two properties of the operator O mentioned above, guarantee the renormalizability of the perturbed theory, (2.2). When $\epsilon \neq 0$, the theory $S(\epsilon)$ is no longer conformal except when O is a truly marginal operator.

Now consider coupling $S(\epsilon)$ to 2d gravity. General coordinate invariance allows us to set two components of metric to fixed values, leaving a single dynamical component. For our purposes it is convenient to choose the light cone gauge:

$$d^2s = dx^+ dx^- + h_{++}(dx^+)^2. \quad (2.3)$$

Here h_{++} is only quantum gravitational dynamical degree of freedom.

As usual, the interaction between the matter system $S(\epsilon)$ and the gravity is through the former's stress tensor.

$$S(\epsilon; h) = S(\epsilon) + \int d^2x h_{++} T_{--}. \quad (2.4)$$

When the operator O has conformal dimension* $\lambda \neq 1$, then the stress tensor acquires a nonvanishing trace. To leading order in ϵ the trace is given by the conservation equation

$$\nabla_+ T_{--} + \partial_- \Theta = 0, \quad (2.5)$$

where

$$\Theta = -2\epsilon (1 - \lambda) O. \quad (2.6)$$

There are two gauge constraints corresponding to the two gauge conditions. Namely,

$$T_{++} = \frac{\delta \Gamma}{\delta h_{--}} = 0, \quad h_{--} = 0 \quad (2.7)$$

and

$$\frac{\delta \Gamma}{\delta h_{+-}} = \Theta - \frac{c}{6} \partial_-^2 h_{++} = 0, \quad h_{+-} = 1. \quad (2.8)$$

Here Γ is the full effective action of 2d gravity [16], whereas c is the Virasoro central charge of the CFT described by S . In the light cone gauge, by definition

$$e^{\frac{i\epsilon}{6}\Gamma} = \langle e^{i \int d^2x h_{++} T_{--}} \rangle_{\text{matt}}. \quad (2.9)$$

From eq. (2.9) one can obtain the equation of motion for h_{++} . To leading order in ϵ one finds

$$\frac{c}{6} \partial_-^3 h_{++} - 2\epsilon(1 - \lambda) \partial_- O = 0. \quad (2.10)$$

The system of eqs. (2.7), (2.8) and (2.10) completely defines the quantum theory (2.4) to leading order in the perturbation. In particular the constraint given by eq. (2.8) leads to the conformal invariance of the theory (2.4).

*This conformal dimension is to be computed by taking into account gravitational dressing, i.e in accordance with the KPZ formula [17].

3 Gravitational dressing of the renormalization group

The renormalization group beta function of the coupling ϵ to leading orders in ϵ is given as follows (see, e.g. [15])

$$\beta = (2 - 2\lambda)\epsilon - \pi C \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (3.11)$$

where the constant C is related to the coefficient of the three point function

$$\langle\langle O(x_1)O(x_2)O(x_3) \rangle\rangle = \frac{C ||O||^2}{x_{12}^{2\lambda} x_{13}^{2\lambda} x_{23}^{2\lambda}} \quad (3.12)$$

with $x_{12} = x_1 - x_2$ etc. Here

$$||O||^2 = \langle\langle O(1)O(0) \rangle\rangle, \quad (3.13)$$

where the double brackets are defined according to

$$\langle\langle \dots \rangle\rangle = \int \mathcal{D}h_{++} \langle \dots e^{i \int d^2x h_{++} T_{--}} \rangle_{\text{matt}}. \quad (3.14)$$

The problem of gravitational dressing has been solved in [15], where the Ward identities for dressed correlation functions have been derived. These Ward identities allow one to compute Green functions of conformal operators in the presence of 2d gravity. In the case under consideration, we need to compute the two and three point functions of the perturbation O in the presence of the quantum gravity.

Following ref. [15], we present the perturbation operator O as follows

$$O(x) = \sum_{AA'} \lambda_{AA'} J_-^A J_+^{A'}, \quad (3.15)$$

where the currents obey the following conservation law

$$(\mathcal{K} + 2)\partial_+ J_-^A = h_{++}\partial_- J_-^A + \lambda(\partial_- h_{++})J_-^A \quad (3.16)$$

with a similar equation existing for $J_+^{A'}$.

The general covariance gives rise to the following Ward identity [16]

$$\langle\langle h_{++}(y)O(x_1)\dots O(x_n) \rangle\rangle = \frac{1}{\mathcal{K} + 2} \sum_i \left[\frac{(y^- - x_i^-)^2}{(y^+ + x_i^+)} \frac{\partial}{\partial x_i^-} + 2\lambda \frac{(y^- - x_i^-)}{(y^+ - x_i^+)} \right] \langle\langle O(x_1)\dots O(x_n) \rangle\rangle, \quad (3.17)$$

where the constant \mathcal{K} is defined below. Eq. (3.16) along with eqs. (3.15) and (3.17) give rise to the Klebanov-Kogan-Polyakov differential equation [15].

At this point the concept of scaling dimension has to be carefully defined. Since we treat the theory $S(\epsilon)$ in perturbation theory, the scaling dimension λ of the perturbation O coincides with its scaling dimension in the theory S coupled to 2d gravity. The formula for anomalous dimensions is [2],[16],[17]:

$$\lambda - \Delta_0 = \frac{\lambda(\lambda - 1)}{\mathcal{K} + 2}, \quad (3.18)$$

where Δ_0 is scaling dimension of O without gravity, whereas the constant \mathcal{K} is given by the formula

$$\mathcal{K} + 2 = \frac{1}{12} (c - 13 + \sqrt{(c - 1)(c - 25)}). \quad (3.19)$$

In the case of the quasimarginal operator O , the KPZ formula (3.18) gives rise to the relation

$$2 - 2\lambda = \frac{\mathcal{K} + 2}{\mathcal{K} + 1} (2 - 2\Delta_0). \quad (3.20)$$

On the other hand, it has been shown that the three point function gets dressed according to the formula [15]

$$C = \frac{\mathcal{K} + 2}{\mathcal{K} + 1} C_0, \quad (3.21)$$

where C_0 is the coefficient of the three point function before 2d gravity was turned on.

Combining eqs. (3.20), (3.21) in formula (3.11), we arrive at the following relation

$$\beta = \frac{\mathcal{K} + 2}{\mathcal{K} + 1} \beta_0, \quad (3.22)$$

with β_0 being the renormalization group beta function computed without gravity. This formula reveals dressing of the renormalization group in the presence of gravity. So far, in the light cone gauge, this effect has been discovered in [15] for marginal perturbation (in the conformal gauge this effect was discussed in [10],[18]). The important point to be made is that the coefficient in front of β_0 in eq. (3.22) stems from the gravitational dressing of correlation functions computed at $\epsilon = 0$. Therefore, this coefficient does not depend on the perturbation parameter ϵ . Hence, it is not going to depend on the scale (t) at least to the leading orders under consideration. Also it is necessary to point out

that eq. (3.22) is a first order equation with respect to t -derivative. Correspondingly the dressed running coupling ϵ can be derived by integrating eq. (3.22). It might be interesting to understand how the same coupling can be obtained from the second order equation discussed in [15],[10].

It is clear from equation (3.22) that all fixed points of the perturbed CFT remain critical points in the presence of 2d gravity at the same values of the coupling constant. However, the flow from the UV critical point to the IR critical point undergoes some changes. In particular, the difference between Virasoro central charges at IR and UV conformal points will be different from the same quantity in the absence of gravity.

4 $c=25$ barrier

The effect of gravitational dressing of the renormalization beta function in the presence of 2d gravity gives rise to a new formula for the difference between the Virasoro central charges at the IR and UV conformal points. This formula can be thought of as a generalization of the Cardy-Ludwig formula [22]. We find that in the presence of the gravity the difference is given as follows

$$\Delta c = c_{IR} - c_{UV} = - \left(\frac{\mathcal{K} + 2}{\mathcal{K} + 1} \right) \frac{y_0^3}{C_0^2} ||O||^2 = \frac{(\mathcal{K} + 2)^2}{(\mathcal{K} + 1)(\mathcal{K} + 3)} \Delta c_0, \quad (4.23)$$

where $y_0 = 2 - 2\Delta_0$, whereas $\Delta c_0 = c_{IR0} - c_{UV0}$ is the difference in the absence of gravity. While $c_{UV} = c_{UV0}$, $c_{IR} \neq c_{IR0}$. Hence, at the IR conformal point the conformal matter must couple to the quantum gravity essentially nonminimally. In other words, along the renormalization group flow the character of interaction between conformal matter and 2d gravity changes drastically. This may give some hints at understanding the issue of strong gravitational coupling.

As one can see there are two singularities emerging at $\mathcal{K} = -1$ and at $\mathcal{K} = -3$ in formula (4.23). The first value of \mathcal{K} corresponds to $c = 25$; whereas $\mathcal{K} = -3$ to $c = 1$. It is a well known fact that Polyakov's quantization of 2d gravity fails when $1 < c < 25$. However there are no physical obstruction for the existence of the quantum gravity coupled to conformal matter with c in this interval. So, there should exist some way to overcoming this problem. Renormalization group flows may shed some light on

this. Indeed, according to Zamolodchikov's c-theorem [23], the Virasoro central charge may decrease along the flow. In particular, the central charge which was slightly larger than 25 at the UV conformal point may become less than 25 at the IR conformal point. Since all critical points must remain fixed points in the presence of 2d gravity there is at least a possibility that flows connecting them exist even if the gravity is turned on. Because the coefficient $(\mathcal{K} + 2)/(\mathcal{K} + 1)$ is very large when c is closed to 25, the difference Δc becomes even larger in the presence of gravity. Therefore, c may become less than 25 along the flow. Of course, the gravity gets changed itself along the flow, however within perturbation theory these changes can be taken into account due to eqs. (2.7), (2.8), (2.10).

In order to justify this conjecture we need a CFT with Virasoro central charge being slightly larger than 25. It turns out that the theory of such a kind is the nonunitary WZNW model on the group manifold G . This model is characterized by *negative* integer level k . In the large $|k|$ limit, the Virasoro central charge of this theory is given as follows

$$c = c_{UV} = \dim G + \frac{c_V(G) \dim G}{|k|} + \mathcal{O}(1/k^2), \quad (4.24)$$

where $c_V(G)$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation of G . Thus, when $\dim G = 25$ we have a desired situation: c is just slightly greater than 25.

Certainly, the nonunitary WZNW model is a highly nontrivial theory. Because it has states with negative norm, its proper definition is quite complicated. To a certain extent this theory is defined by its algebraic properties, i.e by its affine and Virasoro symmetries. We are not going to discuss in this paper all these problems. What we can say, is that there definitely exist unitary representations of the Virasoro symmetry in the nonunitary WZNW model [24]. In particular, there is an operator which satisfies the two properties discussed at the beginning of section 2 and at the same time this operator provides a unitary Virasoro representation with respect to the Mubious invariant vacuum. This operator, in fact, is nothing but the kinetic term of the WZNW model. Thus, one can define an appropriate perturbation on the nonunitary WZNW model [25].

It has been established that the nonunitary WZNW model perturbed by its kinetic

term flows to the unitary WZNW model [25]. The difference Δc_0 is given by [25]

$$\Delta c_0 = \frac{2c_V(G) \dim G}{k} + \mathcal{O}(1/k^2). \quad (4.25)$$

One can estimate the coefficient in the expression for the dressed beta function given by eq. (3.22). One finds

$$\frac{\mathcal{K} + 2}{\mathcal{K} + 1} \approx \sqrt{\frac{|k|}{4c_V(G)}}. \quad (4.26)$$

This gives rise to

$$\Delta c \approx -25 \sqrt{\frac{c_V(G)}{|k|}}. \quad (4.27)$$

Thus, when $|k|$ is very large we obtain

$$c_{IR} = c_{UV} + \Delta c \approx 25 - 25 \sqrt{\frac{c_V(G)}{|k|}} < 25. \quad (4.28)$$

This amounts to a R.G. flow under the $c = 25$ barrier, and so justifies our claim that flows under $c = 25$ can exist. However, based on the above analysis we cannot as yet reveal the way the conformal matter couples to 2d quantum gravity under the barrier. One possibility might be to compute correlation functions under the barrier by using perturbation theory.

Although our conjecture that flows under the $c = 25$ barrier has been justified by consideration of a nonunitary WZNW model only, certainly there may exist other CFT's with appropriate properties near the barrier. Moreover, formula (4.23) holds for any flow between fixed points which are close to each other. Therefore, our conclusion about flows under the $c = 25$ barrier is based only on computations in 2d quantum field theory[†].

As mentioned above, it might be that quantum gravity itself also can be explored under the $c = 25$ barrier by perturbation theory. Equations (2.8) and (2.10) allow one to present h_{++} as

$$h_{++} = J^+ - 2x^- J^0 + (x^-)^2 J^- + \frac{12}{c} \epsilon(1 - \lambda) \frac{\partial_+^2}{\square^2} O. \quad (4.29)$$

[†]In the string cosmology approach to R.G. flows in 2d gravity described in [10], our results would suggest that there should exist cosmological solutions admitting a change of space-time signature. However, at present it is not known whether such solutions actually exist.

Here J^i are the $SL(2, R)$ currents of the quantum gravity when the perturbation is switched off. One can use this representation of h_{++} to compute all Ward identities to leading orders in the perturbation theory. We hope to return to this in a future publication.

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